Title

Sums of Angles
- Equilateral triangles have all angles the same and all sides the same
- Base angles of an isosceles triangle are equal
- Straight line angles add to 180°
- Angles in a triangle add to 180° and angles in a quadrilateral add to 360°

Parallel Line Laws
- Alternate angles are equal
- Corresponding angles are equal
- Same side/co-interior angles add to 360°

Forming Right Angled Triangles
- Construct a perpendicular and use Pythagoras OR SOHCAHTOA
- Symmetry of opposite side lengths of a parallelogram being the same

Dealing With Circles
- 2 diameter lengths are the same!
- 2 radii lengths are the same!
- Radii form isosceles triangles where base angles at the same!

Does making **constructions** help
- No diameter? Create it!
- No centre? Create it!
- No radii? Draw them!
- No triangle or quadrilateral? Create it!

So many circle theorems depend on a centre so don't be afraid to draw it!

Remember to involve the centre and involve radii!

Similar Shapes - Proves All Angles Are The Same
- To prove shapes are similar
  - ✓ Prove all pairs of sides are in ratio

    OR
  - ✓ Prove all angles are the same
    - Prove all sides are in ratio hence similar
    - Call something x and use circle theorems, sum of straight line angles, sum of angles in a shape etc to show that all angles are the same in terms of x

Geometric proofs don't ask you to find the value of the angles, just to prove they are the same.

Congruent Shapes – Proves All Sides Are Same
- To prove all sides are the same
  - ✓ Prove the shapes are congruent
- Proving a triangle is congruent?
  - ✓ Use one of the 4 congruency theorems
- Call something x and use symmetry

Look out for common sides of two shapes as obviously they are the same length!!

When a question doesn't have any lengths then alarm bells should ring: "is this a congruent shapes question?"; "what can I call x"; etc.

Proving angles are the same
Call something x and use circle theorems, straight lines, sum of angles etc. Notice how you are not being asked to find the value of the angles, just to prove they are the same

What if no lengths given?
There is not much else to do other than use similar shapes or congruent shapes. Ask yourself whether similar or congruent shapes?

SSS

SAS – Notice how this angle is in the middle of the 2 sides (formed by the 2 sides) hence we have A in the middle in the name SAS

AA

RHS – We need the hypotenuse, right angle and any side out of the two remaining 2 sides to be the same.

SSA (NOT TRUE!!!!!!!!!!) – Note: Watch out SSA is **NOT** a valid criteria!!!

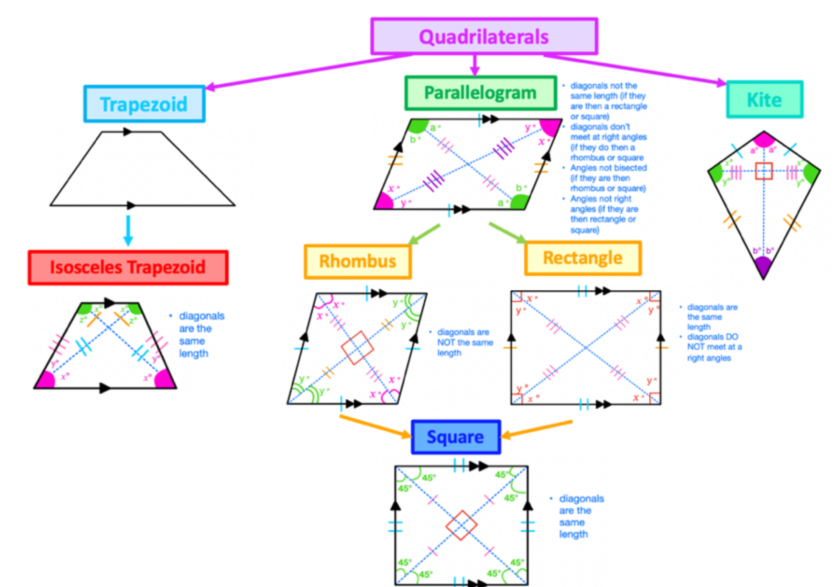
Are constructions necessary?
- No diameter? Create it!
- No centre? Create it!
- No radii? Draw them!

So many circle theorems depend on a centre and radius so don't be afraid to draw them in!

- No triangle or quadrilateral? Create it!
- Do not be afraid to call something x or even something else y

For example, prove that angle CAB is the same as angle ABC. We have to make a construction and call an angle x to get going

2D Shape Properties



The most common properties:
- Parallelograms have opposite sides equal in length
- Opposite angles of a parallelogram are equal

Quadrilaterals	Parallelogram	Rectangle	Rhombus	Square	Trapezium	Isosceles trapezium	Kite
All sides equal			✓	✓			
Opposite sides equal	✓	✓	✓	✓		One pair	2 disjoint pairs of consecutive sides are equal
Opposite sides parallel	✓	✓	✓	✓	One pair	One pair	
Opp. Angles equal	✓	✓	✓	✓		Base angles add to 180°	Only one pair of opposite angles are equal (larger pairs of angles)
4 right angles		✓		✓			
Consecutive angles add to 180°	✓	✓	✓	✓	Non base angles add to 180°	Non base angles add to 180°	
Diagonals equal		✓		✓			
Diagonals are perpendicular			✓	✓			✓
Diagonals bisect each other	✓	✓	✓	✓			
Diagonals bisect pair of opposite angles (bisect vertices)			✓	✓			✓

Circle Theorems

Prove the angle at the centre is double the angle at the circumference

Form radii
- Label x and y (both base angles same since radii form 2 isosceles triangles)

Colour in the remaining angles of the triangle
- Sum of angles in a triangle is 180°: $180-2x$ and $180-2y$
- Angle at point means sum of angles is 360°: Angle at centre $= 360-(180-2x)-(180-2y) = 2x+2y$
- Angle at circumference $x+y$

$2x+2y$ is double $x+y$

Prove the angle in semicircle is a right angle

Form radii
- Label x and y (both base angles same since radii form 2 isosceles triangles)

Colour in the remaining angles of the triangle
- Sum of angles in a triangle is 180°. Bae angles ares $180-2x$ and $180-2y$
- Straight line angle means sum of angles is 180°

$$180-2x+180-2y=180$$
$$-2x-2y=-180$$
$$x+y=90$$

Prove that opposite angles of a cyclic quadrilateral sum to 180°

Draw center and form 2 radii
- Label x and y

Colour in the angles at the centre
- Angle at centre is double angle at the circumference: $2x$ and $2y$
- Angle at point means sum of angles is 360°: $2x+2y=360$

$$x+y=180$$

Prove that angles in the same segment are equal

Draw center and form 2 radii
- Label x

Colour in the angles on the circumference
- Angle at centre is double angle at the circumference: $\frac{1}{2}x$ and $\frac{1}{2}x$

These angles are equal

Prove the alternate segment theorem

Form the diameter AC and CB
- Label x

Colour in the remaining angles in the triangle and the tangent angles
- Sum of angles in a triangle is 180°: $180-90-x=90-x$
- Tangent meets radius at 90°: $90-(90-x)=x$

Hence, $x=x$

With Circles

Example 1

A and B are points on a circle, centre O. BC is tangent to the circle. AOC is a straight line. Angle OAB = x°. Find the size of angle ACB, in terms of x. Give your answer in its simplest form

Form triangle ABO
- Angle ABO=Angle OAB= x° (AOB is an isosceles triangle)
- Angle CBO= 90° (radius meets a tangent at 90°)
- Angle ACB= $180-x-(90+x)=90-2x$ (angles in triangle ABC add to 180°)

Example 2

A, B and C are points on the circumference of a circle, centre O. TA and TB are tangents to the circle. CA=CB. Angle ATB=$2x$°. Show that angle ACB=$(90-x)$°

Form the lines OA and OB (the radii)
- Angle OAT and angle OBT = 90° (tangents meet a radius at 90°)
- Angle AOB= $360-90-90-2x=180-2x$ (angles in a quadrilateral add to 360°)
- Angle ACB= $\frac{180-2x}{2}=90-x$ (angle at the centre is double the angle at the circumference)

Example 3

A, B, C and D are points on the circumference of a circle, centre O. FDE is tangent to the circle. i. Show that $y-x=90$°. Dylan was asked to give to some possible values for x and y. He said x could be 200 and y could be 110, because $200-110=90$°. ii. Is Dylan correct?

Way 1:

Form the line OB
Angle OBD= x (OBD is an isosceles triangle)
Angle BOC= $180-2x$
Angle BCD= $\frac{180-2x}{2}=90-x$ (angle at centre is double angle at circumference)

BADC is a Cyclic quad: $y=180-(90-x)=90+x$

Re-arranging $y=90+x$ gives $y-x=90$

Way 2:

Angle BCD = $180-y$ (cyclic quadrilateral)
Angle BDF= $180-y$ (alternate segment theorem)
$180-y+x=90$ (tangent meets a radius at 90)

Re-arranging gives $y-x=90$

Example 4

P, Q, R and S are points on the circumference of a circle, centre O. APB, BQC, CRD, and DSA are tangents to the circle. ABCD is a kite. Angle PAS= $2x$°. Angle QCR= y°. Find an expression in terms of x and y for the size, in degrees, of the angle POQ.

- Angle APO= 90° (tangents meets a radius at 90°)
- Angle OQC= 90° (tangents meets a radius at 90°)

Form AC which cuts both angles BAD and BCD in half
- Angle PAO= x
- Angle QCO= $\frac{y}{2}$
- Angle AOP= $180-90-x=90-x$ (angles in triangle APO add to 180°)
- Angle COQ= $180-90-\frac{y}{2}=90-\frac{y}{2}$ (angles in triangle OQC add to 180°)
- Angle POQ $=180-(90-x)-\left(90-\frac{y}{2}\right)=x+\frac{y}{2}$ (angles on a straight line add to 180°)

Example 5

ABP and ADQ are tangents to the circle, centre O. C lies on the circumference of the circle. Prove that $y=2x$

All radii are the same length
- Tangent meets a radius at 90°
- Angle DCO = x since base angles of an isosceles triangle are equal
- Angle DOC = $180-x-x=180-2x$ since angles in triangle DOC add to 180°
- Angle OBC = $90-2x$ since tangent meets a radius at 90°
- Angle OCB = $90-2x$ since base angles of an isosceles triangle are equal
- Angle DOC = $180-(90-2x)-(90-2x)=4x$ since angles in triangle OBC add to 180°
- Angle DOB = $360-4x-(180-2x)=180-2x$ since angles at a point add to 360°
- Angle BAD = $180-(180-2x)=2x$

Example 6

D, E, F and S are points on a circle. RST is tangent. The straight line EDT is parallel to FS. DS = DT. Prove that FD is parallel to RST. Use angle DTS as x to help you

- DS = DT hence isosceles triangle hence base angles are equal
- Angle SDT = $180-x-x=180-2x$ since angles in triangle SDT add to 180°
- Angle EDS = $180-(180-2x)=2x$ since angles on a straight line add to 180°
- Angle DFS = angle DST = x since alternate segment theorem
- Angle FDE DFS = angle DFS= x since alternate interior angles
- Angle FDS = x since total angle is $2x$ and angle FDE = x

Angle DTR = angle EDF = x hence corresponding angles are equal thus parallel

Angle FDT + angle DTR $=180-2x+x+x=180$ hence co-interior angles add to 180° thus parallel

Similar Triangles

Example 1

A, B, R and P are four points on a circle with centre O. A, O, R and C are four points on a different circle. The two circles intersect at the points A and R. CPA, CRB and AOB are straight lines. Prove that angle CAB = angle ABC

Triangle ABC is not isosceles or equilateral. Can't use any circle theorems as the question stands.

Form triangle ROB so we can get a triangle and two quadrilaterals, otherwise there isn't very little that we can do. This allows us to use the fact that radii are equal and gives a cyclic quadrilateral to work with.

We don't know any angles so call angle x and relate other angles to it.
Call $\angle CAB = x$°
Note: we could start with any angle as x.

CAOR is a cyclic quadrilateral which means opposite angles add to 180°
CRO is $180-x$

Angles on straight line (CRB) add to 180°
So, $\angle ORB = 180-(180-x)=x$

$\angle CAB = \angle ORB = x$°

Triangle ORB is an isosceles triangle (OB and OR are radii of the blue circle so OR=OB, and by definition of an isosceles triangle the base angles are the same
$\angle OBR = \angle ORB = x = \angle CAB$

$\angle OBR$ is the same angle as $\angle ABC$. hence $\angle OBR = \angle ABC = \angle CAB$

Example 2

In the diagram, P, S and T are points on the circumference of a circle. O is the point such that
- OPS is a straight line
- OT is tangent to the circle

Prove triangle OPT is similar to OTS

- O is same angle in both triangles, so $\angle POT = \angle TOS$
- Alternate segment theorem hence $\angle OTP = \angle TSO$
- So, by AA criterion, triangle OPT is similar to triangle OTS.

Example 3

A, B, D and E are points on a circle. ABC and EDC are straight lines. Prove that the angles in triangle BCD are the same as the angles in triangle ECA

Here we are not given much information (not given any side lengths like usual). But we have a circle, so circle theorems should come to mind. We see a four-sided shape inside a circle which makes it likely that the cyclic quadrilateral theorem is applicable here.

$\angle BCD = \angle ACE$ is common to both triangles which means we already know this angle is the same for both triangles.

Since we don't know any angles the easiest (and only) way to deal with this is to use algebra and start by calling one of the angles x

The diagram below shows the and the numbered order that is best to find them in:

So, we have the following colour pairs which are the same

$\angle BCD = \angle BCD$ (common angle)
$\angle AEC = \angle DBC$ (since we have shown both are equal to $180-x$)
$\angle EAC = \angle BDC$ (since we have shown both are equal to $180-y$)

3 angles in both triangle ΔBCD and ΔECA are the same
Hence $\Delta BCD \cong \Delta ECA$ by AA criterion.

Example 4

Prove that the triangle APC and triangle PBC are similar

Since we do not have any lengths, we need to call something x. Say $\angle BAC = x$°. Since the angles in a triangle (APC) sum to 180°, we can say

$$\angle ACP = 180-90-x = 90-x$$

We are told $\angle ACB = 90$° hence $\angle BCP = 90-(90-x)=x$

Finally,
$$\angle CBA = 180-90-x = 90-x$$

Since
$$\angle CBA = 90-x = \angle ACP, \quad \angle BAC = x = \angle BCP$$

We can utilise AA criterion to say APC ≅ PBC.

Congruent Shapes

PQRS is a quadrilateral.
PQ is parallel to SR. SP is parallel to RQ.
Prove triangle PQS is congruent to RSQ.

- Opposite sides of a parallelogram are equal
- Opposite sides of a parallelogram are equal
- Same length

SSS

Note: AAS and SAS and ASA would have also worked here

ABCD as a quadrilateral. AD=BC. AD is parallel to BC. Prove that triangle ADC is congruent to triangle ABC.

Careful, this is not a parallelogram! We only have 1 pair of parallel lines.

SAS

PQ=PR. S is the midpoint of PQ. T is the midpoint of PR. Prove that triangle QTR is congruent to triangle RSQ.

- The triangles share a common side QR
- QS=TR since midpoints and PQ=PR
- Angle PQR=Angle PRQ (base angles of an isosceles triangle are equal and triangle is isosceles since told PQ=PR)

Congruent by SAS

With Congruent Shapes

Example 1

A, B, C and D are four points on a circle. AEC and DEB are straight lines. Triangle AED is an equilateral triangle. Prove the triangle ABC is congruent to triangle to DCB

Angle AEB= Angle DEC (vertically opposite angles are equal)
Angle EDC=Angle EAB (angles in the same segment are equal)
Angle ABE=Angle DCE (angles in the same segment are equal)

Hence, the triangles have 3 pairs of equal angles therefore similar by AA criterion.

Example 2

A and B are points on a circle, centre O. MAP and NBP are tangents to the circle. Prove that AP=BP

We are basically proving the theorem that tangents a circle from the same point are the same length

$\angle OAP = \angle OBP = 90°$ angle between a tangent and radius

OA=OB since radii of a circle
OP is a common side

Triangles OPB and OAP are congruent by RHS
Hence AP=BP

Example 3

A and B are points on a circle, centre O. BA = BC

Prove that OB bisects angle ABC

AC+OB+OC since all radii of a circle
BA=BC (given)

Triangles AOB and OBC are congruent by SSS
Hence the angles are the same in both triangles

$$\angle OBC = \angle BCO = \angle ABO = \angle OAB = x$$
$$\angle BOC = \angle BOA = 180-2x$$
$$\angle ABO = \angle OBC = x$$

Hence OB bisects ABC

Example 4

AOC and BOD are diameters of a circle, centre O. Prove that triangle ABD and triangle DCA are congruent.

Way 1: ASA

Angle BAD=Angle CDA= 90° (angle in a semicircle is a right angle)
AD is common to both triangles
Angle ABD=Angle ACD (angles in the same segment are equal)

Congruent by AAS

Way 2: RHS

BD=CA (both diameters of a circle)
Angle BAD=Angle CDA= 90° (angle in a semicircle is a right angle)
AD is common to both triangles

ABD and triangle DCA are congruent by RHS

Way 3: SAS

BD=CA (both diameters of a circle)
AD is common to both triangles
Angle ADB=Angle CAD (base angles of the isosceles triangle formed by the radii are equal)

ABD and triangle DCA are congruent by SAS

Example 5

The diagonals shows a quadrilateral XBYA. The diagonals AB and XY intersect at the point M. Given that the area of triangle AXB is equal to the area of triangle AYB. Prove XY is bisected by AB.

First let's think about where we are going with the question i.e. what we need to prove. We need to prove that XM=MY since told that XY is bisected (cut in half) by AB

We are told that the area of triangle AXB is equal to the area of triangle AYB. Since both triangles square a common base, the heights must be equal. Let's form the heights in both of the triangles. We form the heights with the diagonal to make a triangle as we want 2 triangles to use the fact they are congruent later.

If we can prove that triangles GMY and MHX are congruent then we are done since the triangles are identical and hence corresponding side lengths the same
- Angle GMY = angle XMH (vertically opposite angles)
- Angle MGY = angle MHX (right angles)
- The heights (XH and GY) are equal

Triangles GMY and MHX are congruent by AAS

Therefore XM=MY hence XY is bisected by AB

Example 6

ABCD is a square. BEC and DCF are equilateral triangles.
i.Prove that triangle ECD is congruent to triangle BCF.
G is the point such that BEGF is a parallelogram.
ii.Prove that ED =EG

- CD = BC (square)
- FC = CE (both equilateral triangles DCF and BEC all same length since they are sides of a common square)

All the pink and blue lengths are equal since both triangles are equilateral and the square sides being equal links all sides to each other hence all sides (BC, EC, BE CF, DF and CD can all be coloured pink). I will only colour the ones that need for the congruent triangles pink.

- Angle FCB = angle DCE = 90 + 60

SAS

Example 7

ABCD is a parallelogram. ABP and QDC are straight lines. $\angle ADP = \angle CBQ = 90°$
i. Prove that ADP is congruent to triangle CBQ
ii.Explain why AQ is parallel to PC

We want to prove that the orange and green triangles are congruent

i.
- AD = BC (opposite sides of a parallelogram are equal)
- Angles ADP = angle QBC = 90° (given)
- Opposite angles in a parallelogram are equal so angles DAB and BCD are equal

ASA

ii. Triangles ADP and CBQ are congruent which means AP and QC will be equal in length and we know these lines are parallel as this is given in the question. Also AQ = PC. Therefore, APCQ is a parallelogram and opposite sides of a parallelogram are parallel hence AQ is parallel to PC

With Trigonometry

Example 1

A right-angled triangle is formed by the diameters of three semi-circular regions, A, B and as shown in the diagram.

Show that
Area of region A = area of region B + area of region C

Area of A= $\frac{1}{2}\pi\left(\frac{a}{2}\right)^2=\frac{\pi}{8}a^2$

Area of B= $\frac{1}{2}\pi\left(\frac{b}{2}\right)^2=\frac{\pi}{8}b^2$

Area of C= $\frac{1}{2}\pi\left(\frac{c}{2}\right)^2=\frac{\pi}{8}c^2$

$b^2+c^2=a^2$ by Pythagoras
Multiply both sides by $\frac{\pi}{8}$

$$\frac{\pi}{8}b^2+\frac{\pi}{8}c^2=\frac{\pi}{8}a^2$$

Area of B+ Area of C= Area of A

Example 2

The diagram shows three right-angled triangles. Prove that $y=\frac{3}{4}n$

Somehow have to link all 3 triangles since want to link 'n' in the top triangle and 'y' in the bottom triangle

Way 1:

Look at the pink triangle:
$$\frac{n}{b}=\tan 30=\frac{1}{\sqrt{3}} \Rightarrow b=\sqrt{3}n$$

Look at the green triangle:
$$\frac{a}{b}=\cos 30=\frac{\sqrt{3}}{2} \Rightarrow a=\sqrt{3}n\times\frac{\sqrt{3}}{2}=\frac{3n}{2}$$

Look at the blue triangle:
$$\frac{y}{a}=\sin 30=\frac{1}{2} \Rightarrow y=\frac{1}{2}\left(\frac{3n}{2}\right)=\frac{3}{4}n$$

Way 2:

Look at the pink triangle:
$$\frac{n}{c}=\sin 30=\frac{1}{2} \Rightarrow c=2n$$

Using Pythagorean,
$$b^2+n^2=c^2=4n^2 \Rightarrow b=\sqrt{3}n$$

Look at the blue triangle:
$$\frac{y}{a}=\sin 30=\frac{1}{2} \Rightarrow a=2y$$

Look at the green triangle:
$$\frac{2y}{b}=\cos 30=\frac{\sqrt{3}}{2} \Rightarrow y=\frac{b\sqrt{3}}{4}=\sqrt{3}n\times\frac{\sqrt{3}}{4}=\frac{3}{4}n$$

Example 3

The diagram shows a hexagon ABCDEF. ABEF and CBED are congruent parallelograms where AB=BC=x cm. P is the point on AF and Q is the point on CD such that BP=BQ=10 cm. Given that the angle ABC=30°, prove that

$$\cos\angle PBQ = 1-\frac{(2-\sqrt{3})}{200}x^2$$

Look at triangle ABC and use cosine rule
$$AC^2=x^2+x^2-2(x)(x)\cos 30=(2-\sqrt{3})x^2$$

We don't need to square root to find AC since we will be using the squared version in the cosine rule below
Notice that AC=PQ by symmetry so
$$PQ^2=(2-\sqrt{3})x^2$$

Look at triangle BPQ and use cosine rule
$$(2-\sqrt{3})x^2=10^2+10^2-2(10)(10)\cos y$$

So,
$$\cos y=\frac{200-(2-\sqrt{3})x^2}{200}=1-\frac{(2-\sqrt{3})}{200}x^2$$

Hardest Circles

Two circle overlap. A, B and E lie on the circle, centre O. B, C, D and E lie on the other circle AOBC and AED are straight lines. CD = CE
Angle BAE $=x$
i. Give a reason why angle BEA = 90°
ii. Prove than angle DCE $=2x$

i. Angle in a semi-circle is 90°
ii.
- Angle EBA $=180-90-x=90-x$ since angles in triangle BEA add to 180°
- Angle CBE $=180-(90-x)=90+x$ since angles on a straight line add to 180°
- Angle CDE $=180-(90+x)=90-x$ since cyclic quadrilateral CBDE
- Angle DEC $=90-x$ since triangle CDE is isosceles
- Angle DCE $=180-(90-x)-(90-x)=2x$ since angles in triangle DCE add to 180°

A, B, C and D are points on a circle. D, E and F are points on a different circle, centre O. DCE, ADF and BCF are straight lines. Angle DEF $=x$
i. Prove that angle BAD $=2x$
ii. In the case where AB is parallel to DE, work out the size of angle x

i.
- Angle OEF = angle OFE= x since base angles of an isosceles triangle
- Angle EOF $=180-x-x=180-2x$ since angles in triangle FOE add to 180°
- Angle DOB $=180-2x$ since vertical angles are equal
- Angle BAD $=180-(180-2x)=2x$ since cyclic quadrilateral

ii.
- Angle EDF $=180-90-x=90-x$ since angles in triangle add to 180°

If AB and DE parallel then the corresponding angles are equal hence $2x=90-x$
$$3x=90$$
$$x=30$$

Diagram shows two circle touching externally at T. Points X, Y and W lie on the larger circle. RTS is tangent to both circles. XYRZ is tangent to the smaller circle at Z. ZTW is a straight line. $\angle YTR = a$ and $\angle ZTR = b$
i. Give a reason why angle RZT $=b$
ii. Prove that angle XTW = angle YTZ

i. RZ and RT are tangents to a circle and hence the same length. Triangle RZT is therefore isosceles which means the base angles are equal
ii.
- angle TRZ $=180-b-b=180-2b$ since angles in triangle TRZ add to 180°
- angle TRY $=180-(180-2b)=2b$ since angles on a straight line add to 180°
- angle TYR $=180-2b-a$ since angles in triangle TRY add to 180°
- angle XYT $=180-(180-2b-a)=2b+a$ since angles on a straight line add to 180°
- angle YTR = angle TXY (angles in the same segment)
- angle XTY $=180-a-(2b+a)=180-2a-2b$ since angles in triangle add to 180°
- angle XTW $=180-a-b-(180-2a-2b)=a+b$ since angles on a straight line add to 180°